Lecture 2 **2012/10/08**

Outline

1. Schauder Fixed Point Theorem Without Proof
2. Ascoli-Arzela Lemma
3. Inequalities

1.2 Preliminaries:

1. Schauder Fixed Point Theorem

Schauder Fixed Point Theorem. Let E be a Banach space, $D \subset E$ is a closed, bounded, convex set and $f: D \to D \subset E$ is completely continuous. Then f has at least one fixed point x in D.

Remark 2.1 This result ensures the existence of a fixed point only. No uniqueness!

Remark 2.2 D is convex if $x_1, x_2 \in D \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in D,\ 0 \le \lambda \le 1$; f is completely continuous if f is continuous and for any bounded set $D_1 \subset D$, $f(D_1)$ is relatively compact in D; i.e. Any sequence $\{y_n\} \subset f(D_1)$, there exists a convergent subsequence in D.

2. Ascoli-Arzela Lemma

Ascoli-Arzela Lemma Let $F = \{ f: [a,b] \to R^n \}$ be a family of functions on $[a,b]$ such that

- F is equicontinuous;
- F is uniformly bounded

Then the family F has a convergent subsequence $\{g_n\}$, which converges uniformly to $g \in C([a,b])$.

Remark 2.3 F is equicontinuous on $[a,b]$ if $\forall\, \varepsilon > 0$, there exists $\delta > 0$, s.t. for any $f \in F$ and any $t_1, t_2 \in [a,b]$, we have $\| f(t_1) - f(t_2) \| < \varepsilon$ whenever $| t_1 - t_2 | < \delta$;
F is uniformly bounded on $[a,b]$ if there exists $M > 0$ s.t. for any $f \in F$, we have

$\| f(t) \| \le M$, $t \in [a,b]$. .

Remark 2.4 A bounded closed set in $C([a,b])$ is not necessarily compact! However, a (uniformly) bounded set of equicontinuous functions in $C([a,b])$ is compact! This can be regarded as a generalization of Bolzano-Weierstrass theorem in R^n to $C([a,b])$.

Remark 2.5 F is not necessarily countable and uniformly bounded. $F = \{f_j\}$ and $\| f(t) \| \le M(t)$ are acceptable in $[a,b] \subset R$.

Remark 2.6 In Ascoli-Arzela Lemma, if $[a,b] \subset R$ is replaced by $D \subset R^m$ which is compact, the lemma is still true by the similar way to show. Left for homework.

Proof. Step 1. Construction of the desired subsequence.

Take the whole rational numbers $\{r_i\}$ in $[a,b]$. Since $\{f(r_1)\}$ is bounded in R^n, there exists a convergent subsequence $\{f_{1k}(r_1)\}$ $(k \in N^+)$ by Bolzano-Weierstrass theorem, i.e. $\{f_{1k}(t)\}$ converges at $t = r_1$. For $\{f_{1k}(r_2)\}$, which is still bounded in R^n, there exists a convergent subsequence $\{f_{2k}(r_2)\} \subset \{f_{1k}(r_2)\}$ by the same reason. $\{f_{2k}(t)\}$ converges at $t = r_1, r_2$. After similar n steps, we can find following countable subsequences:

$$\begin{array}{ccccc} f_{11}(t) & f_{12}(t) & \cdots & f_{1n}(t) & \cdots \\ f_{21}(t) & f_{22}(t) & \cdots & f_{2n}(t) & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots, \\ f_{n1}(t) & f_{n2}(t) & \cdots & f_{nn}(t) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

where $\{f_{nk}(t)\}$ converges at $t = r_1, r_2, \cdots, r_n$.

Taking the diagonal sequence $g_n(t) = f_{nn}(t)$ $(n \in N^+)$, this sequence converges at any $\{r_i\}$ by the construction because $\{g_n(r_i)\}_{n \ge i} = \{f_{nn}(r_i)\}_{n \ge i}$ is a subsequence of $\{f_{i\,n}(r_i)\}_{n\geq i}$ which converges.

Step 2. It remains to show that $g_n(t)$ is uniformly convergent on $[a,b]$, i.e. $\forall\,\varepsilon>0$, there exists $N\geq 1$ s.t. $n,m\geq N \;\Rightarrow\; \|g_n(t)-g_m(t)\|<\varepsilon$.

Since $g_n(t)$ converges on $\{r_i\}$, we have that for $\forall\,\varepsilon>0$ and any $r_i\in[a,b]$, there exists $N(r_i)$ s.t. $n,\ m>N(r_i) \;\Rightarrow\; \|g_n(r_i)-g_m(r_i)\|<\frac{\varepsilon}{3}$.

By equicontinuity, and for the above given $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ s.t. for any $t_1,t_2\in[a,b]$, $|t_1-t_2|<\delta(\varepsilon) \;\Rightarrow\; \|g_n(t_1)-g_n(t_2)\|<\frac{\varepsilon}{3}$ for all $n\in N^+$.

Taking $O(r_i,\delta(\varepsilon))$, then $\bigcup_{i=1}^{\infty}O(r_i,\delta(\varepsilon))\supseteq[a,b]$. There exists a finite numbers of $O(r_i,\delta(\varepsilon))$ $(i=1,\cdots,p)$ s.t. $\bigcup_{i=1}^{p}O(r_i,\delta(\varepsilon))\supseteq[a,b]$.

Let $N=\max\{N(r_1),N(r_2),\cdots,N(r_p)\}$. Once $n,m\geq N$ and $t\in[a,b]$, there exists one $O(r_{i_0},\delta(\varepsilon))$, $1\leq i_0\leq p$ s.t. $t\in O(r_{i_0},\delta(\varepsilon))$. Then

$$\begin{aligned}\|g_n(t)-g_m(t)\| &\leq \|g_n(t)-g_n(r_{i_0})\|+\|g_n(r_{i_0})-g_m(r_{i_0})\|+\|g_m(r_{i_0})-g_m(t)\|\\ &\leq\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.\end{aligned}$$

This completes the proof. □

3. Inequalities

Gronwall's Inequality. Suppose that $g(t)$, $h(t)\in C([t_0,T])$ with $h(t)\geq 0$ and $C\geq 0$. If

$$g(t)\leq C+\int_{t_0}^{t}g(s)h(s)ds,\quad t\in[t_0,T],$$

then we have

$$g(t)\leq Ce^{\int_{t_0}^{t}h(s)ds},\quad t\in[t_0,T].$$

Proof. Let $u(t)=C+\int_{t_0}^{t}g(s)h(s)ds$, then $u(t)\geq g(t)$, $t\in[t_0,T]$.

$\Rightarrow$ $u(t)$ is differentiable and $u'(t)=g(t)h(t)\leq u(t)h(t)$ by $h(t)\geq 0$. Let

$$v(t) := u'(t) - u(t)h(t) .$$

Then $v(t) \le 0$. We solve the following linear initial problem

$$u'(t) = u(t)h(t) + v(t) \text{, with } v(t_0) = C$$

to obtain

$$u(t) = e^{\int_{t_0}^{t} h(s)ds}[C + \int_{t_0}^{t} v(s)e^{-\int_{t_0}^{s} h(\tau)d\tau} ds],$$

which yields $g(t) \le u(t) \le Ce^{\int_{t_0}^{t} h(s)ds}$. □

Bellman Inequality (Corollary) Suppose that $g(t) \in C([t_0, T])$ and $h \ge 0$ and $C \ge 0$. If

$$g(t) \le C + h\int_{t_0}^{t} g(s)ds \,, \quad t \in [t_0, T] \,,$$

then we have

$$g(t) \le Ce^{h(t-t_0)}, \quad t \in [t_0, T] \,.$$

Remark 2.7 These inequalities have several ways to prove. You explore some others by yourself. The above way takes advantage of some knowledge of differential equations. However, Check carefully, the condition $g(t) \ge 0$ doesn't need in this proof. How to apply these inequalities effectively is one of the most important issues.

Generalized Gronwall's Inequality. Suppose that $g(t)$, $C(t)$, $h(t) \in C([t_0, T])$ with $C(t) \ge 0$ and $h(t) \ge 0$. If

$$g(t) \le C(t) + \int_{t_0}^{t} g(s)h(s)ds \,, \quad t \in [t_0, T] \,,$$

then we have

$$g(t) \le C(t) + \int_{t_0}^{t} h(s)C(s)e^{\int_s^t h(\tau)d\tau} ds \,, \quad t \in [t_0, T] \,.$$

Remark 2.8 The proof of Generalized Gronwall's Inequality is left for homework.